# Combinatorics

## Sequences and Words

* A **sequence** is an ordered list of objects, with repetitions of the same objects allowed (as opposed to a set).
* The objects of a sequence are called **terms**.
* A sequence may be finite:

Or infinite

* The order matters;
* is a different sequence than .
* If all terms of a sequence are from a set , the sequence is a **sequence in** or a **-sequence**.
* For example, is a sequence in .
* It’s also a sequence in , in , in , and in .
* A sequence can also be called a **word** in the alphabet .
* The sequence with possible values for each . Then:

#### Corollary:

* Let .
* Then there are sequences of length in .

Exercise:

How many 3-letter words can be formed with the English alphabet?



## Permutations

* A sequence in which all terms are distinct is called a **permutation**.
* If , a sequence of length of all distinct objects is called a **permutation of objects taken at a time**.
* If , we just say **permutations of objects.**

Exercise:

Let . The following words in are permutations of 6 objects taken 3 at a time.

The following words in are permutations of 6 objects.

* There are Permutations of objects taken k at a time.
* Notice that .
* This has a shorter notation called **“falling factorial”** , which is also used for .
* When , we have:

, and when

Exercise:

Let .



### Theorem:

* For all , there are permutations of objects taken at a time.

#### Proof:

If , there is no way to permute objects at a time, so the answer must be zero.

If , there are choices for the 1st element, then choices for the 2nd, etc.

So the possibilities are:

#### Corollary:

* For all , there are permutations of objects.

## Counting Strategies

* Consider the problem, “how many sequences satisfy a certain set of properties?”
* We use counting strategy to answer this question methodically.
* For a sequence of length , use empty slots:



* Fill each slot one at a time, with the number of possible values for each term, given the restrictions of the properties.



* By multiplication rule, there are possible sequences.

Exercise:

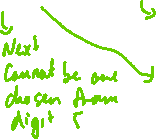
There are 2 highways from Brisbane to Sydney, and 3 highways from Sydney to Adelaide. How many different round trips from Brisbane to Adelaide via Sydney are there? How many are there without taking the same highway twice?



* You don’t necessarily have to start with the 1st position.
* Start where it’s most convenient.

Exercise:

How many 5-digit odd numbers with no repeated digits are there?



* Sometimes, we need to break a problem up into subproblems.

Exercise:

How many 5-digit even numbers with no repeated digits are there?



### Required Adjacency

* For a required adjacency, treat the adjacency as a single object, then multiple by the number of arrangements of the adjacency.

Exercise:

Three single people and a married couple are to be seated in a row of chairs. In how many ways can it be done such that the spouses sit together?



### Forbidden Adjacency

* For a forbidden adjacency, calculate it as a required adjacency, and then subtract from the total possible arrangements.

Exercise:

In how many ways can you align a cow, a goat, a fox, and a chicken such that the fox and the chicken are not next to each other.



## Binomial Coefficients

* Recall the power set of .
* Another notation for is . This is because of the following.

### Theorem:

* Let
* Then has subsets, i.e.

#### Proof:

With induction.

a) Let , then , and , so

b) Let , suppose and . Define

.

The subsets of Y are those that contain y, and those that do not. Those that do not are exactly the subsets of X, of which there are . Those that do contain y are of the form , where , so there are exactly of those too. Therefore, .

* Let . For every , we denote by the number of subsets of with elements.
* The symbol Is read **“ choose ”** or **“the BINOMIAL COEFFICIENT of order ”**
* Some are obvious:

, since the only subset of cardinality 0 is .

, since is the only subset of with elements.

If , then , as it’s impossible to have a subset of with cardinality larger than that of .

### Theorem:

* For all ,

#### Proof:

For , we’ve seen that , and .

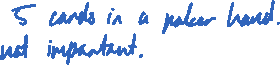
Let . Recall that the number of permutations of objects taken at a time is .

This number can be obtained by taking all combinations of elements and ordering the elements in each combination, which can be done in ways. Thus,

* The symbol is also denoted by , the number of combinations of objects taken at a time.

Exercise:

How many different poker hands are there?



### Theorem:

* For all

#### Proof:

### Theorem:

* For all ,

#### Proof:



## The Binomial Theorem

* Motivation:
* In how many ways can 3 red marbles and 4 blue marbles be arranged in a row? (Or a more practical example: how many binary words are there with 3 zeros and 4 ones?). The multiplication rule isn’t very helpful here; there are too many cases. However, considering the 7 slots:



Notice that once you choose slots for the red marbles, the placement of the blue ones is automatic. So the question is, how many ways are there to choose 3 of the 7 slots? We know the answer is . Similarly, if you choose 4 slots for the blue marbles first, there are ways to do it. The answer is the same, because

### Theorem:

* The number of words of length consisting of letters of one sort, and , letters of a second sort is:
* Consider the binomial expansion

which is the sum of all words of length 2 in the alphabet .

Similarly,

is the sum of all words of length 3 in the alphabet .

By simplifying, we get the familiar formulae:

* The binomial theorem below is a formula for the coefficients of binomial expansion to any power in .

### Theorem (Binomial Theorem):

* For all ,

#### Proof:

The case is easily verified by hand. For , the expansion of is (before simplification) the sum of all words of length in the alphabet .

The number of such words that consist of and is by the previous theorem.

The binomial theorem as written gives the expansion in ascending powers of x:

Equivalently, it can be written in reverse:

We can substitute values for and to obtain identities.

Exercise:

Let . Then the binomial theorem gives:

Exercise:

Let . Then the binomial theorem gives:

* Sometimes, a useful trick is to use the fact that .

Exercise:

Simplify .

A:

Exercise:

Simplify

A: